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Classical and Quantal Einstein Relativistic Two-Particle Systems

T. Aaberge

Département de Physique Théorique, Université de Genève, 1211 Genève 4, Switzerland

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We present formalisms for the description of two-particle systems of classical and of quantal Einstein relativistic particles. For each case the presentation follows a standard scheme. We define the phase-space, the observables, and the action of the kinematical symmetry group in the center-of-mass representation. We then discuss some of the elementary features of the description of two-particle systems in order to be able to interpret the objects considered. In particular, we show that the description of the free particles conforms to standard relativistic kinematics. As an application we discuss the system consisting of two charged particles interacting via the Coulomb field.

1. INTRODUCTION

The theory for the description of conservative systems of two Einstein relativistic particles presented in this paper is a compromise between the space-time approach (Pearle, 1968; Aghassi, Roman, and Santilli, 1970; Horwitz and Piron, 1973; Aaberge, 1975; Reuse, 1979) and the momentum-space approach (Wigner, 1939; Newton and Wigner, 1949; Bakamjian and Thomas, 1953; Fong and Sucher, 1964; Coester, 1965). The choice of phase space that we have made has been suggested by the space-time approach. However, this a priori choice, which is necessary in order to describe particles interacting via fields localized in space-time, gives a phase space which is too big to allow for an interpretation as state space, i.e., for each of its points to represent a state of the system. To accommodate for this defect, we apply constraints serving as constraints to the "mass shell." In this way we incorporate the Einstein relativistic kinematics, and reproduce important features of the description of two-particle systems in the momentum space approach.

We apply the notational conventions $i, j, k \in \{1,2,3\}$, $\alpha, \beta, \gamma, \delta, \mu, \nu, \ldots \{0,1,2,3\}$, $(q^i) = (q^1, q^2, q^3)$, $(q^\mu) = (q^0, q^1, q^2, q^3)$, $(q^i)^2 = q^i q^i = q^{1^2} + q^{2^2} + q^{3^2}$ and $q^\mu q_\mu = q^2 - (q^i)^2$. Moreover, to distinguish between functions (operators) and values of functions (operators) we use a caret.

2. THE SYSTEM OF TWO CLASSICAL PARTICLES

The system of two classical Einstein relativistic particles is defined in a center-of-mass representation "diagonalizing" the space-time coordinates. We then discuss briefly the canonical coordinates "diagonalizing" the mass-defect observable of the center of mass. In this representation the conjugate observable to the momentum is the classical Newton-Wigner position observable.

The Hamilton dynamics is introduced via the Cartan one-form and constraints are defined to which the dynamics is subjected.

We describe the motion of the center of mass and show that the scattering system is asymptotically equivalent to the system of two free particles. The kinematical basis for the construction of the present theory has been discussed in Aaberge (1982).

2.1. Definition of the Two-Particle System

Definition 2.1. A system of two classical Einstein relativistic particles of kinematical masses m_1 and m_2 ($m_1 \geqslant m_2 > 0$) is associated with (i) the space:

$$\Omega = \Gamma \times \mathbb{R} = \{(P^\mu, Q^\mu, p^\mu, q^\mu, t) \in T^*(N \times \mathbb{R}^3) \times \mathbb{R}\}$$

$$N = \Big\{ (P^\mu, p^0) \in \mathbb{R}^5 | (P^0 + Mc)^2 - (P^i)^2 > 0, P^0 > -Mc$$

$$\& \frac{m_1}{M} \left[(P^0 + Mc)^2 - (P^i)^2 \right]^{1/2} > p^0$$

$$> -\frac{m_2}{M} \left[(P^0 + Mc)^2 - (P^i)^2 \right]^{1/2}, \qquad M = m_1 + m_2 \Big\}$$

and the phase space (Γ, ω), with

$$\omega = dP_\mu \wedge dQ^\mu + dp_\mu \wedge dq^\mu$$

(ii) the kinematical symmetry group $SO(3,1) \underset{s}{\times} \mathbb{R}^4$ being represented by the action

$$
\begin{aligned}
P^\mu &\mapsto \Lambda(\theta^i, u^i)^\mu_\nu P^\nu + Mv^\mu(u^i) \\
Q^\mu &\mapsto \Lambda(\theta^i, u^i)^\mu_\nu Q^\nu + tv^\mu(u^i) + a^\mu \\
&\quad + p_\alpha \Lambda^{-1}\big(\theta^i_w(P^\mu, \Lambda(\theta^i, u^i))\big)^\alpha_\beta \Lambda(\theta^i, u^i)^\mu_\gamma \partial_{P_\gamma} \Lambda\big(\theta^i_w(P^\mu, \Lambda(\theta^i, u^i))\big)^\beta_\nu q^\nu \\
p^\mu &\mapsto \Lambda\big(\theta^i_w(P^\mu, \Lambda(\theta^i, u^i))\big)^\mu_\nu p^\nu \\
q^\mu &\mapsto \Lambda\big(\theta^i_w(P^\mu, \Lambda(\theta^i, u^i))\big)^\mu_\nu q^\nu \\
t &\mapsto t
\end{aligned}
$$

where Λ denote the usual Lorentz transformations, and θ^i_w are the functions defined by the identity[1]

$$
\Lambda\big(\theta^i_w(P^\mu, \Lambda(\theta^i, u^i))\big)^\mu_\nu = L^{-1}\big(\Lambda(\theta^i, u^i)^\mu_\nu P^\nu + Mv^\mu(u^i)\big)^\mu_\alpha \Lambda(\theta^i, u^i)^\alpha_\beta L(P^\mu)^\beta_\nu
$$

with

$$
L(P^\mu)^\mu_\nu = \Lambda\left(0^i, \frac{P^i}{P^0 + Mc}\right)
$$

(iii) the observables P^μ, Q^μ, ΔM, and X^μ describing the center of mass, the observables p^μ and q^μ of the internal system, and the time t, being represented by the functions

$$
\begin{aligned}
\hat{P}^\mu(P^\mu, Q^\mu, p^\mu, q^\mu, t) &= P^\mu \\
\hat{Q}^\mu(P^\mu, Q^\mu, p^\mu, q^\mu, t) &= Q^\mu \\
\hat{p}^\mu(P^\mu, Q^\mu, p^\mu, q^\mu, t) &= p^\mu \\
\hat{q}^\mu(P^\mu, Q^\mu, p^\mu, q^\mu, t) &= q^\mu \\
\hat{t}(P^\mu, Q^\mu, p^\mu, q^\mu, t) &= t
\end{aligned}
$$

$$
\Delta\hat{M} \circ \Phi(P^\mu, Q^\mu, p^\mu, q^\mu, t) = \frac{1}{c}\left((P^0 + Mc)^2 - (P^i)^2\right)^{1/2} - M
$$

$$
\hat{X}^\mu \circ \Phi(P^\mu, Q^\mu, p^\mu, q^\mu, t) = \left(\frac{\left[(P^0 + Mc)^2 - (P^i)^2\right]^{1/2}}{P^0 + Mc} Q^0 c, Q^i - \frac{P^i}{P^0 + Mc} Q^0\right)
$$

[1]Notice that $\Lambda(\theta^i_w(\ldots))$ is a rotation.

It might be worthwhile to notice that the observable X^i of the above definition is the classical realization of the Newton–Wigner position observable.

Theorem. Let $(Y^\mu) = (\Delta M, P^i)$; then the map

$$\Phi: \Gamma \to \Gamma, \qquad (P^\mu, Q^\mu, p^\mu, q^\mu) \mapsto (Y^\mu, X^\mu, p^\mu, q^\mu)$$

is of the form $\Phi = T^*\varphi$, where

$$\varphi: N \times \mathbb{R}^3 \to N \times \mathbb{R}^3, \qquad (P^\mu, p^\mu) \mapsto (Y^\mu, p^\mu) \tag{1}$$

is a diffeomorphism.

Proof. The proof follows by verification of the definition. ■

Corollary. The map $\Phi: \Gamma \to \Gamma$ is a symplectomorphism of (Γ, ω).

Proof. The proof follows from the observation that $\omega = -d(Q^\mu\, dP_\mu + q^\mu\, dp_\mu)$. ■

In the coordinates $(Y^\mu, X^\mu, p^\mu, q^\mu)$, Γ is characterized by

$$\Gamma = \{(Y^\mu, X^\mu, p^\mu, q^\mu) \in \mathbb{R}^{16} |\ \Delta M > -M$$
$$\& \ (m_1/M)(\Delta M + M)c > p^0 > -(m_2/M)(\Delta M + M)c\}$$

The action of $SO(3,1) \underset{s}{\times} \mathbb{R}^4$ is given by

$$\Delta M \mapsto \Delta M$$

$$P^i \mapsto P^i + \frac{\gamma^2}{\gamma+1}\frac{(u^j p^j)}{c}u^i + \frac{u^i}{c}\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2} = P'^i$$

$$X^\mu \mapsto \Sigma(\Delta M, P^i, \Lambda(\theta^i, u^i))^\mu_{\ \nu} X^\nu + \Pi^\mu(\Delta M, P^i, u^i)t$$
$$+ p_\alpha \Lambda^{-1}(\theta^i_w(\cdot))^\alpha_{\ \beta}\Sigma(\Delta M, P^i, \Lambda(\theta^i, u^i))^\mu_{\ \gamma}\partial_{Y_\gamma}\Lambda(\theta^i_w(\cdot))^\beta_{\ \nu} q^\nu$$
$$+ \left(\frac{(\Delta M + M)c}{\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2}}a^0, a^i\right.$$
$$\left. - \frac{P^i}{\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2}}a^0\right)$$

for

$$\Sigma(\Delta M, P^i, \Lambda(\theta^i, 0^i))^\mu_\nu = \Lambda(\theta^i, 0^i)$$

$$\Sigma(\Delta M, P^i, \Lambda(0^i, u^i))^0_0 = 1$$

$$\Sigma(\Delta M, P^i, \Lambda(0^i, u^i))^0_i = \frac{(\Delta M + M) c u^i}{\left[(P^i) + (\Delta M + M) c)\right]^{1/2}}$$

$$\Sigma(\Delta M, P^i, \Lambda(0^i, u^i))^i_0 = 0$$

$$\Sigma(\Delta M, P^i, \Lambda(0^i, u^i))^i_j = \delta^i_j + \frac{1}{\gamma+1}\frac{u^i u^j}{c^2} - \frac{P'^i u^j}{c\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2}}$$

$$\Pi^\mu(\Delta M, P^i, u^i) = \left(\frac{(\Delta M + M)(\gamma - 1) c}{\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2}}, u^i \right.$$
$$\left. - \frac{P'^i(\gamma - 1) c}{\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2}} \right)$$

Notice that X^i transforms independently of X^0 and q^0.

Finally, the observables P^μ, Q^μ, ΔM, and X^μ are realized by the functions

$$\hat{P}^\mu \circ \Phi^{-1}(Y^\mu, X^\mu, p^\mu, q^\mu, t) = \left(\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2} - Mc, P^i \right)$$

$$\hat{Q}^\mu \circ \Phi^{-1}(Y^\mu, X^\mu, p^\mu, q^\mu, t) = \left(\frac{\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2}}{(\Delta M + M) c} X^0, X^i \right.$$
$$\left. + \frac{P^i}{(\Delta M + M) c} X^0 \right)$$

$$\Delta\hat{M}(Y^\mu, X^\mu, p^\mu, q^\mu, t) = \Delta M$$

$$\hat{X}^\mu(Y^\mu, X^\mu, p^\mu, q^\mu, t) = X^\mu$$

2.2. The Dynamics. Let $\omega_{\mathcal{H}}: T_*\Omega \to \mathbb{R}$ be the 1-form

$$\omega_{\mathcal{H}} = P_\mu \, dQ^\mu + p_\mu \, dq^\mu - Mc \, dQ^0 - \hat{\mathcal{H}}(P^\mu, Q^\mu, p^\mu, q^\mu, t) \, dt$$

and let $\xi_{\mathcal{H}}$ be a vector field on Ω such that

$$i_{\xi_{\mathcal{H}}} d\omega_{\mathcal{H}} = 0$$

then, $\xi_{\mathcal{H}}$ is unique and given by (Cartan, 1971)

$$\xi_{\mathcal{H}} = \partial_{P_\alpha} \hat{\mathcal{H}}(\cdot)\, \partial_{Q^\alpha} - \partial_{Q^\alpha} \hat{\mathcal{H}}(\cdot)\, \partial_{P_\alpha} + \partial_{p_\alpha} \hat{\mathcal{H}}(\cdot)\, \partial_{q^\alpha} - \partial_{q^\alpha} \hat{\mathcal{H}}(\cdot)\, \partial_{p_\alpha} + \partial_t$$

The function $\hat{\mathcal{H}}$, the Hamiltonian of the system, is assumed to contain the information about the dynamics of the system in the sense that a curve c: $[t_1, t_2] \to \Omega$ describes a possible evolution only if it satisfies the equations of motion

$$\dot{c} = \xi \circ c$$

Further restrictions are imposed on the dynamics by the assumption that the Hamiltonian is of the form

$$\hat{\mathcal{H}}(P^\mu, Q^\mu, p^\mu, q^\mu, t) = \frac{P^\mu P^\mu}{2M} + \hat{h}(p^\mu, q^i) \tag{2}$$

where for reasons of Lorentz covariance, the internal Hamiltonian $\hat{h}$ is assumed to be rotation invariant; and moreover, that the possible initial conditions are subjected to the restrictions imposed by the constraints

$$\hat{\alpha} = \Delta \hat{M} c + \frac{\Delta \hat{M}^2}{2Mc} - \frac{1}{c}\hat{h} = P^0 - \frac{1}{c}\hat{\mathcal{H}} = 0$$

$$\hat{\beta} = p^0 - \frac{1}{c}\left(1 - 4\frac{m}{M}\right)^{1/2} \frac{\hat{h}}{(1 + 2h/Mc^2)^{1/2}} = 0 \tag{3}$$

Remark. The functions $\hat{\alpha}$ and $\hat{\beta}$ are constants of motion $\xi_{\mathcal{H}}(\hat{\alpha}) = 0$ and $\xi_{\mathcal{H}}(\hat{\beta}) = 0$. The constraints $\hat{\alpha} = 0$ and $\hat{\beta} = 0$ are thus satisfied during an evolution if they are satisfied on the initial conditions.

The internal energy spectrum is defined to be the range of the internal Hamiltonian $\hat{h}$ restricted to the submanifold

$$\Omega^{(\beta=0)} = \{(P^\mu, Q^\mu, p^\mu, q^\mu, t) \in \Omega \,|\, \hat{\beta} = 0\}$$

Similarly, the total energy spectrum is defined to be the range of $\hat{\mathcal{H}}$ restricted to

$$\Omega^{(\alpha=0,\beta=0)} = \{(P^\mu, Q^\mu, p^\mu, q^\mu, t) \in \Omega \,|\, \hat{\alpha} = 0 \;\&\; \hat{\beta} = 0\}$$

2.3. Characteristic Features of the Description. According to the definition of the last paragraph, a model of a conservative system of two mutually interacting particles and no external field is completely defined by giving an internal Hamiltonian $\hat{h}(p^\mu, q^i)$. The center of mass of the system in an internal state of energy e appears as a free Einstein relativistic particle of kinematical mass M and internal energy e, being associated with the state space

$$\Omega^{(\beta=0,h=e)} = \{(P^\mu, Q^\mu, p^\mu, q^\mu, t) \in \Omega^{(\beta=0)} \,|\, \hat{h} = e\}$$

In the coordinates (Y^μ, X^μ), the Hamiltonian (2) takes the form

$$\hat{\mathcal{H}} \circ \Phi^{-1}(\cdot) = c\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2} - (\Delta M + M)c^2 - \frac{\Delta M^2}{2M} c^2 + \hat{h}(p^\mu, q^i)$$

The equations of motions for the observables ΔM, P^i and X^μ are thus

$$\Delta \dot{M} = 0$$

$$\dot{P}^i = 0$$

$$\dot{X}^0 = \frac{\Delta M + M}{M} c^2 - \frac{(\Delta M + M)c^3}{\left((P^i)^2 + (\Delta M + M)^2 c^2\right)^{1/2}}$$

$$\dot{X}^i = \frac{P^i c}{\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2}}$$

Parametrizing the solutions by the velocity u^i, we get

$$\Delta M = M\left[(1 + 2e/Mc)^{1/2} - 1\right]$$

$$P^i = (\Delta M + M)\gamma u^i, \qquad \gamma = \left[1 - (u^i)^2/c^2\right]^{-1/2}$$

$$X^0 = \left(\frac{\gamma - 1}{\gamma} + \frac{\Delta M}{M}\right) c^2 t + \frac{1}{\gamma} Q_0^0 c$$

$$X^i = u^i t + Q_0^i - u^i Q_0^0$$

$$P^\mu = (\Delta M + M)((\gamma - 1)c, \gamma u^i) + (\Delta Mc, 0^i)$$

$$Q^\mu = ([(1 + \Delta M/M)\gamma - 1]c, (1 + \Delta M/M)\gamma u^i)t + Q_0^\mu$$

$$E = \frac{P^\mu P^\mu}{2M} + e = (\Delta M + M)\gamma c^2 - Mc^2$$

where we have applied the constraint $\alpha = 0$ to define ΔM as a function of the internal energy e.

When the internal space associated with the chosen internal energy e is trivial, i.e., when

$$\Omega^{(\beta=0, h=e)} \cong \{(P^\mu, Q^\mu, t) \in \mathbb{R}^9 | (P^0 + Mc)^2 - (P^i)^2 > 0, P^0 > -Mc\}$$

then X^μ transforms by

$$X^\mu \to \Sigma(\cdot)^\mu_\nu X^\nu + \Pi^\mu(\cdot)t \tag{4}$$

under a homogeneous Lorentz transformation.

To interpret the notion of position expressed by the observable X^i, let $X'^i = X_0'^i$ denote the position of a particle at rest. Then by (4),

$$X'^i \mapsto X^i = u^i t + X_0'^i - \frac{\gamma}{\gamma+1} \frac{(X_0'^j u^j)}{c^2} u^i = u^i t + X_0^i$$

is the position in a frame moving with velocity K^i. An interpretation of X^i appears from the observation that

$$X_{0\|}^i = X_{0\|}'^i \left[1 - (u^i)^2/c^2\right]^{1/2}$$

$$X_{0\perp}^i = X_{0\perp}'^i$$

i.e., for a system without internal structure, X^i transforms by a Lorentz–Fitzgerald contraction under the Lorentz transformations (O^i, u^i).

Another important feature of this theory concerns the description of the asymptotic states of a scattering system. Let $\hat{h}$ be the internal Hamiltonian of a system of two particles possessing scattering, i.e., such that the study of the asymptotic system is equivalent to the study of the system of two free particles, for which the Hamiltonian is

$$\hat{\mathcal{K}}(\cdot) = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m}$$

Let $\Omega(\mathcal{H})$ be defined by

$$\Omega(\mathcal{H}) = \left\{ \Omega \mid \frac{m_1\left[(P^0+Mc)^2-(P^i)^2\right]-2m_2Mh}{2M\left[(P^0+Mc)^2-(P^i)^2\right]^{1/2}} > p^0 > -\frac{m_2\left[(P^0+Mc)^2-(P^i)^2\right]-2m_1Mh}{2M\left[(P^0+Mc)^2-(P^i)^2\right]^{1/2}} \right\}$$

then

$$\Omega(\mathcal{H}) = T^*(M\times M)\times\mathbb{R}$$

In fact, consider the individual particle observables p_1^μ, q_1^μ, p_2^μ, and q_2^μ defined by

$$p_1^\mu \circ \Psi(P^\mu, Q^\mu, p^\mu, q^\mu, t) = \frac{m_1}{M}P^\mu - L(P^\mu)^\mu_\nu p^\nu$$

$$q_1^\mu \circ \Psi(P^\mu, Q^\mu, p^\mu, q^\mu, t) = Q^\mu + p_\alpha L^{-1}(P^\mu)^\alpha_\beta\, \partial_{P_\mu} L(P^\mu)^\beta_\nu q^\nu - \frac{m_2}{M}L(P^\mu)^\mu_\nu q^\nu$$

$$p_2^\mu \circ \Psi(P^\mu, Q^\mu, p^\mu, q^\mu, t) = \frac{m_2}{M}P^\mu + L(P^\mu)^\mu_\nu p^\nu$$

$$q_2^\mu \circ \Psi(P^\mu, Q^\mu, p^\mu, q^\mu, t) = Q^\mu + p_\alpha L^{-1}(P^\mu)^\alpha_\beta\, \partial_{P_\mu} L(P^\mu)^\beta_\nu q^\nu + \frac{m_1}{M}L(P^\mu)^\mu_\nu q^\nu$$

then, the map

$$\Psi: T^*(M\times M) \to T^*(M\times M)$$
$$(P^\mu, Q^\mu, p^\mu, q^\mu) \mapsto (p^\mu, q^\mu, p^\mu, q^\mu) \qquad (5)$$

for

$$M = \left\{ (p^\mu)\in\mathbb{R}^4 \mid (p^0+mc)^2-(P^i)^2>0 \;\&\; p^0 > -mc \right\}$$

is a symplectomorphism; in fact,

$$\Psi = T^*\psi \qquad \text{where } \psi: M\to M,\ (P^\mu, p^\mu)\mapsto(p_1^\mu, p_2^\mu)$$

thus

$$Q_\mu\, dP^\mu + q_\mu\, dp^\mu = q_{1_\mu}\, dp_1^\mu + q_{2_\mu}\, dp_2^\mu$$

The action (Definition 2.1) of $SO(3,1) \underset{s}{\times} \mathbb{R}^4$ can be transported to the new coordinates. It reads

$$p_\cdot^\mu \mapsto \Lambda(\theta^i, u^i)^\mu_\nu p_\cdot^\nu + m.v^\mu(u^i)$$

$$q_\cdot^\mu \mapsto \Lambda(\theta^i, u^i)^\mu_\nu q_\cdot^\nu + tv^\mu(u^i)$$

Moreover, the observables P^μ, Q^μ, p^μ, and q^μ are represented by the functions

$$\hat{P}^\mu \circ \Psi^{-1}(p^\mu, q^\mu, p^\mu, q^\mu, t) = p_1^\mu + p_2^\mu$$

$$\hat{Q}^\mu \circ \Psi^{-1}(p^\mu, q^\mu, p^\mu, q^\mu, t) = \frac{1}{M}(m_1 q_1^\mu + m_2 q_2^\mu) + \frac{1}{M}(m_1 p_{2\alpha} - m_2 p_{1\alpha}) L(p_1^\mu + p_2^\mu)^\alpha_\beta \times \partial_{p_\mu} L^{-1}(p_1^\mu + p_2^\mu)^\beta_\nu (q_2^\nu - q_1^\nu)$$

$$\hat{p}^\mu \circ \Psi^{-1}(p^\mu, q^\mu, p^\mu, q^\mu, t) = \frac{1}{M} L^{-1}(p_1^\mu + p_2^\mu)^\mu_\nu (m_1 p_2^\nu - m_2 p^\nu)$$

$$\hat{q}^\mu \circ \Psi^{-1}(p^\mu, q^\mu, p^\mu, q^\mu, t) = L^{-1}(p_1^\mu + p_2^\mu)^\mu_\nu (q_2^\nu - q_1^\nu)$$

Thus, we have shown that q^μ is the relative space-time position in the center-of-mass frame of reference. We have moreover, shown that modulo the restriction to $\Omega(\mathcal{H})$, the two-particle system without interaction appears as a system of two free particles. In fact, the constraints $\alpha = 0$ and $\beta = 0$ imply the constraints $\alpha_1 = 0$ and $\alpha_2 = 0$ for

$$\alpha_i = p_i^0 - \frac{p_i^\mu p_{i\mu}}{2m_i}$$

Because of the constraints, the restriction to $\Omega(\mathcal{H})$ is however, inessential. The way of showing this is to introduce constraints $\alpha = 0$ and $\beta = 0$ on Ω and thus to construct the Bakamjian–Thomas center-of-mass representation of the system of two free classical "Wigner particles" (Bakamjian and Thomas, 1953).

2.4. The Bakamjian–Thomas Theory. Let

$$\mathcal{H}(\cdot) = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m} \qquad \hat{h}(\cdot) = \frac{p^\mu p_\mu}{2m}$$

the diffeomorphism

$$\theta: N \times \mathbb{R}^3 \to N \times \mathbb{R}^3$$
$$(P^\mu, p^\mu) \mapsto (\alpha, P^i, \beta, p^i) = (\tilde{Y}^\mu, y^\mu)$$

where α and β are defined by (3), then induces a symplectomorphism

$$\Theta = T^*\theta: T^*(N \times \mathbb{R}^3) \to T^*(N \times \mathbb{R}^3)$$
$$(P^\mu, Q^\mu, p^\mu, q^\mu) \mapsto (\tilde{Y}^\mu, \tilde{X}^\mu, y^\mu, x^\mu)$$

It turns out that $\tilde{X}^\mu$ is given by

$$\hat{\tilde{X}}^0(P^\mu, Q^\mu, p^\mu, q^\mu, t) = \frac{Mc^2}{P^0 + Mc} Q^0$$

$$\hat{\tilde{X}}^i(P^\mu, Q^\mu, p^\mu, q^\mu, t) = X^i(P^\mu, Q^\mu, p^\mu, q^\mu, t) = Q^i - \frac{P^i}{P^0 + Mc} Q^0$$

while x^μ is given as a function of P^μ, p^μ, and q^μ for which we will not give the explicit expressions.

The action (Definition 2.1) of $SO(3,1) \times_s \mathbb{R}^4$ can be transported to these new coordinates. The result is

$$\alpha \mapsto \alpha$$

$$P^i \mapsto P^i + \frac{\gamma^2}{\gamma+1}\frac{(u^j P^j)}{c^2} u^i + \gamma \frac{u^i}{c}\left[(P^j)^2 + M^2c^2 + 2M\alpha c - \frac{M}{m} p^\mu p_\mu\right]^{1/2}$$

$$\beta \mapsto \beta$$

$$p^i \mapsto \Lambda(\theta_w^i(\cdot))^i_j p^j$$

$$x^0 \mapsto x^0$$

$$x^i \mapsto \Lambda(\theta_w^i(\cdot))^i_j x^j$$

for α, P^i, β, p^i, x^0, and x^i. We will not give explicitly the action on $\tilde{X}^\mu$. We only notice that X^i transforms independently of $\tilde{X}^0$ and x^0.

Denote by $\hat{\alpha}^{-1}(0) \cap \hat{\beta}^{-1}(0)$ the intersection of the inverse images of 0 by $\hat{\alpha}$ and $\hat{\beta}$ in Ω:

$$\hat{\alpha}^{-1}(0) \cap \hat{\beta}^{-1}(0) = \{(P^i, \tilde{X}^\mu, p^i, x, t) \in \mathbb{R}^{15}\}$$

Moreover, let $\Omega_0 = \hat{\alpha}^{-1}(0) \cap \hat{\beta}^{-1}(0)/\sim$ be the quotient obtained by identifying two points in $\hat{\alpha}^{-1}(0) \cap \hat{\beta}^{-1}(0)$ if they differ only in the value of the coordinates $\tilde{X}^0$ and x^0. Then,

$$\Omega_0 = \{(P^i, X^i, p^i, x^i, t) \in \mathbb{R}^{13}\} = \Gamma_0 \times \mathbb{R}$$

and Ω_0 carries a well-defined action of $SO(3,1) \underset{s}{\times} \mathbb{R}^4$; in fact, the two operations by means of which Ω_0 is defined are Lorentz invariant. It can also be shown that $\Gamma_0 = \mathbb{R}^{12}$ possesses a Lorentz-invariant symplectic structure

$$dP_i \wedge dX^i + dp_i \wedge dx^i$$

inherited from (Γ, ω).

The resulting theory, the Bakamjian–Thomas theory, describes completely the system of two free particles. The Hamiltonian takes the form

$$\mathcal{H}_0(\cdot) = c\left[(P^i)^2 + \left(\Delta\hat{M}_0(\cdot) + M\right)^2 c^2\right]^{1/2} - Mc^2$$

with

$$\Delta\hat{M}_0(\cdot) = c\left[\left(p^i\right)^2 + m_1^2c^2\right]^{1/2} + c\left[\left(p^i\right)^2 + m_2^2c^2\right]^{1/2} - Mc^2$$

Moreover, one can define an individual-particle representation, in which the system appears as two free classical "Wigner particles" (Bakamjian and Thomas, 1953). It is also possible to use this theory to describe particles in mutual interaction, but only for interaction fields which are localized in "the sense of Newton–Wigner."

2.5. The Galilean Limit. The theory for the description of two classical Galilei relativistic particles can be obtained by (i) taking the limit $c \to \infty$, (ii) and then getting rid of redundant states. According to the first step, a system of two classical Galilei relativistic particles is a priori associated with (i) the state space

$$\Omega_\infty = \Gamma_\infty \times \mathbb{R} = \{(P^\mu, Q^\mu, p^\mu, q^\mu, t) \in \mathbb{R}\}$$

(ii) the kinematical symmetry group $SO(3) \underset{s}{\times} \mathbb{R}^7$ being represented by the action

$$(P^0, P^i) \mapsto \left(P^0, \Lambda(\theta^i)^i_j P^j + Mu^i \right)$$

$$(Q^0, Q^i) \mapsto \left(Q^0 + a^0, \Lambda(\theta^i)^i_j Q^j + tu^i + a^i \right)$$

$$(p^0, p^i) \mapsto \left(p^0, \Lambda(\theta^i)^i_j p^j \right)$$

$$(q^0, q^i) \mapsto \left(q^0, \Lambda(\theta^i)^i_j q^j \right)$$

(iii) the observables P^μ, Q^μ, p^μ, q^μ, and t defined as in Definition 2.1. (Notice that $\Delta M \equiv 0$, x^0 is not defined, and $X^i = Q^i$ in the Galilean limit.)

A given model has a Galilean limit if

$$\lim_{c \to \infty} \hat{\mathcal{H}}(P^\mu, Q^\mu, p^\mu, q^\mu, t) = \frac{P^\mu P_\mu}{2M} + h_\infty(p^\mu, q^i)$$

exists. For such a system the constraints $\alpha = 0$ and $\beta = 0$ imply that $P^0 = 0$ and $p^0 = 0$ in the Galilean limit. The observables Q^0 and q^0 then become redundant and can be discarded. Thus, identifying two states differing only in the values taken by Q^0 and q^0, we can conclude the following:

Definition. A system of two classical Galilei relativistic particles of masses m_1 and m_2 is characterized by (i) the state space

$$\Omega_G = \Gamma_G \times \mathbb{R} = \{(P^i, Q^i, p^i, q^i, t) \in \mathbb{R}^{13}\}$$

and the phase space

$$\left(\Gamma_G, dP_i \wedge dQ^i + dp_i \wedge dq^i \right)$$

(ii) the kinematical symmetry group $SO(3) \times_s \mathbb{R}^6$ being represented by the action given above; (iii) the observables P^i, Q^i, p^i, q^i and t is being represented by functions on Γ_G in the obvious way.

The Galilean limit of the symplectomorphism (5) induces a symplectomorphism $\Gamma_G \to \Gamma_G$, whose inverse is the usual "barycentric map."

3. THE SYSTEM OF TWO QUANTAL PARTICLES

In the present chapter we develop a theory for the description of two quantal Einstein relativistic particles in mutual interaction, in parallel with the corresponding classical theory. From a general point of view the classical and quantum theories are different realizations of a set of imprimitivity systems defining the action of the restricted inhomogeneous Lorentz group on the observables, and the construction can be done by means of Mackey's theory of induced representations (Piron, 1976; Mackey, 1968). The existence and uniqueness of the construction are then assured by the imprimitivity theorem.

3.1. Definition of the System

Definition 3.1. The system of two quantal Einstein relativistic particles of kinematical masses m_1 and m_2 $(m_1 \geqslant m_2 > 0)$ and spins s_1 and s_2 $(s_1, s_2 \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\})$ is by assumption associated with (i) the state space $\{H_t | t \in \mathbb{R}\}$ where each H_t is a Hilbert space isomorphic to

$$L^2\left(N \times \mathbb{R}^3, \mathbb{C}^{2s_1+1} \otimes \mathbb{C}^{2s_2+1}; d^4P\, d^4p\right)$$

i.e., the Hilbert space of functions

$$f: N \times \mathbb{R}^3 \to \mathbb{C}^{2s_1+1} \otimes \mathbb{C}^{2s_2+1}$$

such that

$$\int_{N \times \mathbb{R}^3} \left(f(P^\mu, p^\mu), f(P^\mu, p^\mu)\right) d^4P\, d^4p < \infty$$

$(,)$ is the canonical Hermitian scalar product on $\mathbb{C}^{2s_1+1} \otimes \mathbb{C}^{2s_2+1}$;

(ii) the kinematical symmetry group $SO(3,1) \times \mathbb{R}^4$ being represented by the unitary projective representation

$$\begin{aligned} \left(U(\Lambda(\theta^i, u^i), a^\mu) f\right)_t (P^\mu, p^\mu) &= \exp\left(-\frac{i}{\hbar} P_\mu a^\mu + \frac{i}{\hbar} P_\mu v^\mu(-u^i) t\right) \\ &\times D\left(\Lambda\left(\varphi_w^i(P^\mu, \Lambda(\theta^i, u^i))\right)\right) f_t\left(\Lambda^{-1}(\theta^i, u^i)^\mu_\nu p^\nu + M v^\mu(-u^i),\right. \\ &\qquad \left.\Lambda^{-1}\left(\varphi_w^i(P^\mu, \Lambda(\theta^i, u^i))\right)^\mu_\nu p^\nu\right) \end{aligned}$$

for

$$\Lambda\left(\varphi_w^i(P^\mu, \Lambda(\theta^i, u^i)\right) = \Lambda^{-1}\left(\theta_w^i(P^\mu, \Lambda^{-1}(\theta^i, u^i)\right) \qquad \text{(Definition 2.1)}$$

and

$$D = D^{(s_1)} \otimes D^{(s_1)}$$

the tensor product of two irreducible unitary projective representations of spins s_1 and s_2 of $SO(3)$ on $\mathbb{C}^{2s_1+1}$ and $\mathbb{C}^{2s_2+1}$;

(iii) the observables $P^\mu, \Delta M, X^i$ describing the center of mass, the observations p^μ and q^μ describing the internal system, and the time t. These observables are by assumption represented by the self-adjoint operators

$$(\hat{P}^\mu f)_t(P^\mu, p^\mu) = P^\mu f_t(P^\mu, p^\mu)$$

$$\left(\Delta\hat{M}_\varphi f\right)_t(P^\mu, p^\mu) = \left(\frac{1}{c}\left[(P^0 + Mc)^2 - (P^i)^2\right]^{1/2} - M\right)f_t(P^\mu, p^\mu)$$

$$\left(\hat{X}^i_\varphi f\right)_t(P^\mu, p^\mu) = i\hbar\left[\partial_{P_i} + \frac{P_i}{P^0 + Mc}\partial_{P^0} - \frac{1}{2}\frac{P^i}{(P^0 + Mc)^2}\right]f_t(P^\mu, p^\mu)$$

$$(\hat{p}^\mu f)_t(P^\mu, p^\mu) = p^\mu f_t(P^\mu, p^\mu)$$

$$(\hat{q}^\mu f)_t(P^\mu, p^\mu) = i\hbar\partial_{p_\mu} f_t(P^\mu, p^\mu)$$

$$(\hat{t}f)_t(P^\mu, p^\mu) = tf_t(P^\mu, p^\mu)$$

In addition we can define observables S_1^i and S_2^i of internal spin and let them be represented by the generators $\hat{S}_1^i$ and $\hat{S}_2^i$ of $\hat{D}^{(s_1)}$ and $\hat{D}^{(s_2)}$.

Remark. The observables Q^0 and X^0 have no representatives in this quantum theory. This is a consequence of the well-known fact that the symmetric operator $i\partial/\partial x$ defined in $L^2(\langle 0, \infty\rangle; dx)$ has no self-adjoint extensions.

A representation which is useful for the discussion of models is defined by the unitary transformation

$$\hat{F}_\varphi\colon L^2(N \times \mathbb{R}^3, \mathbb{C}^{2s_1+1} \otimes \mathbb{C}^{2s_2+1}; d^4P\, d^4p)$$

$$\to L^2\left(N \times \mathbb{R}^3, \mathbb{C}^{2s_1+1} \otimes \mathbb{C}^{2s_2+1}; \frac{c\,d\Delta M\, d^3P\, dp^0\, d^3q}{\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2}}\right)$$

$$f(P^\mu, p^\mu) \mapsto g(\Delta M, P^i, p^0, q^i)$$

$$= \left(\frac{1}{\hbar^{1/2}}\right)^3 \int_{\mathbb{R}^3} (f \circ \varphi^{-1})(\Delta M, P^i, p^\mu) \exp\left[\frac{i}{\hbar}(p^i q^i)\right] d^3p \qquad (6)$$

being a composite of a Fourier transformation and the isometry induced by the diffeomorphism φ: $N\times\mathbb{R}\to N\times\mathbb{R}$ (1). In this representation the observables P^μ, ΔM, X^i, p^μ, and q^μ are represented by

$$(\hat{P}_{\varphi^{-1}}g)(\Delta M, P^i, p^0, q^i) = \left(\left[(P^i)^2+(\Delta M+M)^2c^2\right]^{1/2} - Mc, P^i\right)g(\Delta M, P^i, p^0, q^i)$$

$$(\Delta\hat{M}g)(\Delta M, P^i, p^0, q^i) = \Delta M g(\Delta M, P^i, p^0, q^i)$$

$$(\hat{X}^i g)(\Delta M, P^i, p^0, q^i) = i\hbar\left[\partial_{P_i} - \frac{1}{2}\frac{P^i}{(P^i)^2+(\Delta M+M)^2c^2}\right] \times g(\Delta M, P^i, p_0, q_i)$$

$$(\hat{p}^\mu g)(\Delta M, P^i, p^0, q^i) = (p^0, -i\hbar\partial_{q_i})g(\Delta M, P^i, p^0, q^i)$$

$$(\hat{q}^\mu g)(\Delta M, P^i, p^0, q^i) = (-i\hbar\partial_{p^0}, q^i)g(\Delta M, P^i, p^0, q^i)$$

3.2. The Dynamics. The evolution of a quantal system is by assumption given by a family of unitary operators (Piron, 1976)

$$V_t(\tau): H_t \to H_{t+\tau}$$

induced by a permutation of the real line

$$t \mapsto t+\tau$$

i.e.,

$$V_{t+\tau_1}(\tau_2)V_t(\tau_1) = V_t(\tau_1+\tau_2)$$

Under suitable technical conditions this is equivalent to the Schrödinger equation

$$i\hbar\partial_t f_t = \hat{\mathcal{H}}_t f_t$$

where $\hat{\mathcal{H}}_t$ is a self-adjoint operator on H_t, the Hamiltonian of the system.

For a system of two quantal Einstein relativistic particles in mutual interaction the Hamiltonian is supposed to be of the form

$$\hat{\mathcal{H}} = \frac{P^\mu P_\mu}{2M} + \hat{h}$$

where the internal Hamiltonian $\hat{h}$ is given as a rotation invariant function of $\hat{p}^\mu$, $\hat{q}^i$ and the spins only.

Let α and β be defined by

$$\hat{\alpha} = \hat{P}^0 - \frac{1}{c}\hat{\mathcal{H}}$$

$$\hat{\beta} = k^0 - \frac{1}{c}\left(1 - 4\frac{m}{M}\right)^{1/2} \frac{\hat{h}}{\left(1 + 2\hat{h}/Mc^2\right)^{1/2}}$$

Evidently $\hat{\mathcal{H}}$, $\hat{h}$, $\hat{\alpha}$, and $\hat{\beta}$ commute amongst each others, and we can consider the decomposition of $\hat{\mathcal{H}}$ and $\hat{h}$ according to $\hat{\alpha}$ and $\hat{\beta}$, i.e.,

$$\hat{\mathcal{H}} = \int_{\mathrm{sp}(\alpha,\beta)} \mathcal{H}^{(\alpha,\beta)}\, d\mu(\alpha,\beta)$$

$$\hat{h} = \int_{\mathrm{sp}(\alpha,\beta)} \hat{h}^{(\alpha,\beta)}\, d\mu(\alpha,\beta)$$

in

$$H = \int_{\mathrm{sp}(\alpha,\beta)} H^{(\alpha,\beta)}\, d\mu(\alpha,\beta)$$

By assumption, the total and internal energy spectra of the system are associated with the spectra of $\hat{\mathcal{H}}^{(0,0)}$ and $\hat{h}^{(0,0)}$ in $H^{(0,0)}$. We will express this by saying that the system satisfies the constraints

$$\text{“}\alpha = 0\text{”} \quad \text{and} \quad \text{“}\beta = 0\text{”} \tag{7}$$

Remark. The above definition of the energy spectra is slightly delicate in the sense that it is without meaning in a strict Hilbert-space language. In fact, the operators $\hat{\alpha}$ and $\hat{\beta}$ have purely continuous spectra and $\hat{\mathcal{H}}^{(0,0)}$, $\hat{h}^{(0,0)}$, and $H^{(0,0)}$ therefore do not exist. To overcome this difficulty one will have to consider a formulation in terms of rigged Hilbert-space (Gel'fand triple), i.e., work with generalized eigenfunctions.

3.3. Characteristic Features of the Description. Because of the constraints, the center of mass of the system in an internal bound state of energy $e^{(0)}$ and total angular momentum j appears as a free Wigner particle of rest mass

$$M'(e^{(0)}) = M + \Delta M(e^{(0)}) = M\left(1 + 2e^{(0)}/Mc^2\right)^{1/2}$$

and spin j; i.e., if the only degeneracy of the internal bound state spectrum of $\hat{h}$ is that due to rotation invariance, the center of mass is completely described by the carrier space of an irreducible unitary projective representation $(M'(e^{(0)}), j)$ of $SO(3,1) \underset{s}{\times} \mathbb{R}^4$, for each internal bound state $(e^{(0)}, j)$. Moreover, the asymptotic states of a scattering system is given by the tensor product of two one-particle spaces. This last statement can be proved for the case of no interaction, which is equivalent to the asymptotic scattering system. Thus, let the internal Hamiltonian be given by

$$(\hat{h}f)(P^\mu, p^\mu) = \frac{p^\mu p_\mu}{2m} f(P^\mu, p^\mu)$$

and let H_p denote the subspace of H on which

$$\frac{m_1\left[(P^0 + Mc)^2 - (P^i)^2\right] - 2m_2 M\hat{h}}{2M\left[(P^0 + Mc)^2 - (P^i)^2\right]^{1/2}} > p^0 > -\frac{m_2\left[(P^0 + Mc)^2 - (P^i)^2\right] - 2m_1 M\hat{h}}{2M\left[(P^0 + Mc)^2 - (P^i)^2\right]^{1/2}}$$

then,

$$H_p = L^2(M \times M, \mathbb{C}^{2s_1+1} \otimes \mathbb{C}^{2s_2+1}; d^4P\, d^4p)$$

We denote by $\hat{V}_\psi$ the isometry induced by the diffeomorphism ψ: $M \times M \to M \times M$ (3),

$$\begin{aligned}\hat{V}_\psi &: L^2(M \times M, \mathbb{C}^{2s_1+1} \otimes \mathbb{C}^{2s_2+1}; d^4P\, d^4p) \\ &\to L^2(M \times M, \mathbb{C}^{2s_1+1} \otimes \mathbb{C}^{2s_2+1}; d^4p_1\, d^4p_2) \\ &\cong L^2(M, \mathbb{C}^{2s_1+1}; d^4p_1) \otimes L^2(M, \mathbb{C}^{2s_2+1}; d^2p_2)\end{aligned}$$

$$\begin{aligned}(\hat{V}_\psi f)(P^\mu, p^\mu) = g(p_1^\mu, p_2^\mu) &= \left[J_\psi(p_1^\mu, p_2^\mu)\right]^{-1/2} \\ &\times \hat{D}^{(s_1)}\left(\Lambda\left(\varphi_w^i(p_1^\mu, L(P^\mu))\right)\right) \\ &\otimes \hat{D}^{(s_2)}\left(\Lambda\left(\varphi_w^i(p_2^\mu, L(P^\mu))\right)\right) f \circ \psi^{-1}(p_1^\mu, p_2^\mu)\end{aligned}$$

where J_ψ is the Jacobian determinant of ψ.

The image of the representation $\hat{U}$ of $SO(3,1) \underset{s}{\times} \mathbb{R}^4$ on H_p, is the form $\hat{U}^1 \otimes \hat{U}^2$, with

$$\left(\hat{U}^n(\Lambda(\theta^i, u^i), a^u)g\right)_t(p_n^\mu) = \exp\left(-\frac{i}{\hbar} p_n^\mu a_\mu + \frac{i}{\hbar} p_n^\mu v_\mu(-u^i)t\right)$$
$$\times \hat{D}^{(s_n)}\left[\Lambda\left(\varphi_w^i\left(p_n^\mu, \Lambda(\theta^i, u^i)\right)\right)\right]$$
$$\times g_t\left(\Lambda^{-1}(\theta^i, u^i)_\nu^\mu p_n^\nu + m_n v^\mu(-u^i)\right)$$

being a projective unitary representation of $SO(3,1) \underset{s}{\times} \mathbb{R}^4$ in

$$L^2\left(M, \mathbb{C}^{2s_n+1}; d^4 p_n\right)$$

For the spin-0 case, the proof of this assertion is straightforward (see classical case). To extend the proof to the case of nontrivial spins, we must show that

$$\hat{D}^{(s_n)}\left(\Lambda\left(\varphi_w^i\left(p_n, L(P^\mu)\right)\right)\right) \hat{D}^{(s_n)}\left(\Lambda\left(\varphi_w^i\left(P^\mu, \Lambda(\theta^i, u^i)\right)\right)\right)$$
$$\times \hat{D}^{(s_n)^{-1}}\left(\Lambda\left(\varphi_w^i\left(\Lambda^{-1}(\theta^i, u^i)_\nu^\mu p_n^\nu + m_n v^\mu(-u^i),\right.\right.\right.$$
$$\left.\left.\left.L\left(\Lambda^{-1}(\theta^i, u^i)_\nu^\mu P^\mu + M v^\mu(-u^i)\right)\right)\right)\right) = D^{(s_n)}\left(\Lambda\left(\varphi_w^i\left(p_n, \Lambda(\theta^i, u^i)\right)\right)\right)$$

Since $\hat{D}^{(s_n)}$ is a representation of $SO(3)$, the computation can be done with the $\Lambda(\varphi_w)s$. Thus, introducing the definition of $\Lambda(\varphi_w(\cdot))$ and using a simplified notation, we find that

$$\Lambda\left(\varphi_w^i(p, L(P))\right)\Lambda\left(\varphi_w^i(P, \Lambda)\right)\Lambda^{-1}\left(\varphi_w^i\left(\Lambda^{-1}p, L^{-1}(\Lambda^{-1}P)\right)\right)$$
$$= \Lambda\left(\varphi_w^i(p, L(P))\right)\Lambda\left(\varphi_w^i(P, \Lambda)\right)$$
$$\times L^{-1}\left(L^{-1}(\Lambda^{-1}P)\Lambda^{-1}p\right)\right)L^{-1}(\Lambda^{-1}P)L(\Lambda^{-1}p)$$
$$= \Lambda\left(\varphi_w^i(p, L(P))\right)\Lambda\left(\varphi_w^i(P, \Lambda)\right)$$
$$\times L^{-1}\left(\Lambda^{-1}\left(\varphi_w^i(P, \Lambda)\right)L^{-1}(P)p\right)L^{-1}(\Lambda^{-1}P)L(\Lambda^{-1}p)$$
$$= \Lambda\left(\varphi_w^i(p, L(P))\right)L^{-1}\left(L^{-1}(P)p\right)\Lambda\left(\varphi_w^i(P, \Lambda)\right)L^{-1}(\Lambda^{-1}P)L^{-1}p)$$
$$= L^{-1}(p)L(P)L\left(L^{-1}(P)p\right)L^{-1}\left(L^{-1}(P)p\right)$$
$$\times L^{-1}(P)\Lambda L(\Lambda^{-1}P)L^{-1}(\Lambda^{-1}P)L(\Lambda^{-1}p)$$
$$= L^{-1}(p)\Lambda L(\Lambda^{-1}p) = \Lambda\left(\varphi_w^i(p, \Lambda)\right)$$

In the above computation $\Lambda^{-1}p$ stands for $\Lambda^{-1}(\theta^i, u^i)^\mu{}_\nu p^\nu + mv^\mu(-u^i)$ etc; moreover, we have applied the identity

$$\Lambda(\theta^i)\Lambda(u^i)\Lambda^{-1}(\theta^i) = \Lambda\big(\Lambda(\theta^i)^i_j u^j\big)$$

We have thus constructed the individual particle representation; in fact, the observables of four-momentum p_n^μ, mass defect Δm_n and position x_n of particle n, are represented by the self-adjoint operators

$$(\hat{p}_n^\mu g)(p_n^\mu) = p_n^\mu g(p_n^\mu)$$

$$(\Delta \hat{m}_{n\varphi} g)(p_n^\mu) = \left\{\frac{1}{c}\left[(p_n^0 + mc)^2 - (p^i)^2\right]^{1/2} - m\right\} g(p_n^\mu)$$

$$(\hat{x}_{n\varphi} g)(p_n^\mu) = i\hbar\left[\partial_{p_{ni}} + \frac{p^i}{p_n^0 + m_n c}\partial_{p_n^0} - \frac{1}{2}\frac{p^i}{(p_n^0 + m_n c)^2}\right] g(p_n^\mu)$$

on

$$L^2(M, \mathbb{C}^{2s_n+1}; d^4p_n)$$

Moreover, the Hamiltonian $\hat{\mathcal{H}}$ reads in this representation

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_1 \otimes I_1 + I_2 \otimes \hat{\mathcal{H}}_2$$

for

$$(\hat{\mathcal{H}}_n g)(p_n^\mu) = \frac{p_n^\mu p_{n_\mu}}{2m_n} g(p_n^\mu)$$

while the constraints (7) implies the Lorentz-invariant constraints

$$\text{“}\alpha_1 = 0\text{”} \quad \text{and} \quad \text{“}\alpha_2 = 0\text{”}$$

with

$$\hat{\alpha}_n = p_n^0 - \frac{1}{c}\hat{\mathcal{H}}_n$$

Finally, if one introduces the constraints one obtain two free Wigner particles of (effective) masses m_1 and m_2 (i.e., $\Delta m = 0$) and spins s_1 and s_2. Accordingly, modulo the (inessential) restriction to H_p we have shown that the description of a scattering of two particles in this formalism incorporates the usual Einstein relativistic kinematical conservation laws.

Remark. As for the classical case, the restriction to H_p is inessential, because of the constraints. In fact, one may introduce the constraints first to construct the reduced phase space, and from them go to the individual particle representation.

4. THE COULOMB SYSTEM

The following model of the system of two electrically charged particles of spin 0 presents a simple but nontrivial application of the theory outlined above. The model is constructed according to the standard assumption that the particles interact via the Coulomb field.

We determine the internal energy spectrum of the model and discuss some aspects of the description. In view of its approximate character, the model, considered as a model of hydrogenlike or exotic atoms, is acceptable from a phenomenological point of view.

4.1. The Model. The internal Hamiltonian $\hat{h}$ describing the system of two particles of "spin 0" interacting via the Coulomb field is by assumption of the form

$$\hat{h} = \frac{(\hat{p}^i)^2}{2m} - \frac{\left\{\hat{p}^0 - (e_1 e_2/c)\left[(\hat{q}^i)^2\right]^{-1/2}\right\}^2}{2m} + \frac{e_1 e_2}{\left[(\hat{q}^i)^2\right]^{1/2}} \tag{8}$$

where $m = m_1 m_2/(m_1 + m_2)$, and e_1 and e_2 denotes the charges of the two particles.

The problem we will consider is to determine the internal energy spectrum associated with $\hat{h}$. For this it is sufficient to study the families $\hat{h}(p^0)$, $\hat{\alpha}(\Delta M, p^0)$, and $\hat{\beta}(p^0)$ appearing in the decomposition (6), i.e., $\hat{h}(p^0)$, $\hat{\alpha}(\Delta M, p^0)$, and $\hat{\beta}(p^0)$ considered as operators on $L^2(\mathbb{R}^3, d^3q) = L^2(\mathbb{R}^3 \setminus \{0\}, d^3q)$ for each admissible value of ΔM, p^0.

Let σ: $\mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$ be the diffeomorphism defining the spherical coordinates (r, θ, ϕ). σ induces the isometry[2]

$$\begin{aligned} V_\sigma\colon L^2(\mathbb{R}^3; d^3q) &\to L^2(\mathbb{R}^+ \times S^2; \sin\theta\, dr\, d\theta\, d\phi) \\ &\cong L^2(\mathbb{R}^+; dr) \otimes L^2(S^2; \sin\theta\, d\theta\, d\phi) \\ &\cong L^2\left(\mathbb{R}^+, L^2(S^2; \sin\theta\, d\theta\, d\phi); dr\right) \\ (V_\sigma f)(q^1, q^2, q^3) &= r(f \circ \sigma^{-1})(r, \theta, \phi) = \psi(r, \theta, \phi) \end{aligned}$$

[2]We consider only one chart.

Denoting by $\hat{h}_{\sigma^{-1}}(p^0)$ the image of $\hat{h}(p^0)$ under $\hat{V}_\sigma$, we find that

$$[\hat{h}_{\sigma^{-1}}(p^0)\psi](r,\theta,\phi) = \left[-\frac{\hbar^2}{2m}\partial_r^2 + \frac{(l^i)^2(\theta,\phi)-(e_1e_2/c)^2}{2mr^2} + \left(1+\frac{p^0}{mc}\right)\frac{e_1e_2}{r} - \frac{p^{02}}{2m}\right]\psi(r,\theta,\phi)$$

where

$$(\hat{l}^i\psi)(r,\theta,\phi) = l^i(\theta,\phi)\psi(r,\theta,\phi)$$

are the angular momentum operators.

Let $\hat{U}$ be a unitary transformation defining a spectral representation of $(\hat{l}^i)^2$, i.e.,

$$\hat{U}(\hat{l}^i)^2\hat{U}^{-1} = \sum_{l=0}^{\infty} \hbar^2 l(l+1)\hat{P}_l$$

on

$$L^2\left(\mathbb{R}^+;\ \sum_{l=0}\mathbb{C}^{2l+1};\ dr\right)$$

where $\hat{P}_l$ denote the projection onto the spectral subspace of angular momentum l,

$$P_l\colon L^2\left(\mathbb{R}^+,\ \sum_{l=0}^{\infty}\mathbb{C}^{2l+1};\ dr\right) \to L^2(\mathbb{R}^+,\mathbb{C}^{2l+1};\ dr)$$

Since $\hat{h}_{\sigma^{-1}}(p^0)$ is invariant under rotations and thus commutes with $(\hat{l}^i)^2$ it decomposes accordingly,

$$\hat{U}\hat{h}_{\sigma^{-1}}(p^0)\hat{U}^{-1} = \sum_{l=0}^{\infty}(\hat{h}_{\sigma^{-1}})_l(p^0)\hat{P}_l$$

where each partial wave operator $(\hat{h}_{\sigma^{-1}})_l(p^0)$ is defined on $L^2(\mathbb{R}^4, \mathbb{C}^{2l+1}; dr)$ by

$$((h_{\sigma^{-1}})_l(p^0)g)_l(r) = \left[-\frac{\hbar^2}{2m}\partial_r^2 + \frac{\hbar^2 l(l+1)-(e_1e_2/c)^2}{2mr^2} + \left(1+\frac{p^0}{mc}\right)\frac{e_1e_2}{r} - \frac{p^{02}}{2m}\right]g_l(r)$$

Similarly, we obtain $(\hat{\alpha}_{\sigma^{-1}})_l(\Delta M, p^0)$ and $(\hat{\beta}_{\sigma^{-1}})_l(p^0)$.

From Metz (1964) we can conclude that (i) $(\hat{h}_{\sigma^{-1}})_l(p^0)$ is self-adjoint if

$$(l+1/2)^2-(e_1e_2/\hbar c)^2 \geqslant 1$$

(ii) for

$$0 \leqslant (l+1/2)^2-(e_1e_2/\hbar c)^2 \leqslant 1$$

there exists a class of self-adjoint extensions for $(\hat{h}_{\sigma-1})_l(p^0)$. A regularization method selects the extension belonging to the boundary condition

$$g_l(r) \sim r^{[(l+1/2)^2-(e_1e_2/\hbar c)^2]^{1/2}+1/2} \qquad \text{for } r \to 0$$

(iii) for

$$(l+1/2)^2-(e_1e_2/\hbar c)^2 < 0$$

there exists a class of self-adjoint extensions of $(\hat{h}_{\sigma^{-1}})_l(p^0)$, but a regularization method does not lead to a definite extension.

In the following we assume that

$$-1/2 \leqslant e_1e_2/\hbar c < 0$$

Thus $(\hat{h}_{\sigma-1})_l(p^0)$ (i) is self-adjoint for $l \geqslant 1$ and (ii) has a class of self-adjoint extensions for $l=0$; we will consider the regular extension. Notice that these choices define a self-adjoint extension of $\hat{h}$.

The spectrum of $(\hat{h}_{\sigma-1})_l(p^0)$ is then determined by the "eigenvalue equation"

$$\left[(\hat{h}_{\sigma^{-1}})_l(p^0)-e_l(p^0)\hat{I}\right]g_l(r)=0 \tag{9}$$

with the boundary condition

$$g_l(0)=0$$

4.2. The Spectrum of the Internal Hamiltonian. The spectrum of $h_l(p^0)$ is

$$\left\{ -\frac{1}{2}\frac{(1+p^0/mc)^2mc^2}{\left\{n-l-1/2+\left[(l+1/2)^2-\alpha^2\right]^{1/2}\right\}^2}-\frac{p^{02}}{2m}\right|$$

$$\left.\alpha=-\frac{e_1e_2}{\hbar c};\ n=1,2,\ldots;\ =0,1,\ldots\right\} \cup \left[-\frac{p^{02}}{2m},\infty\right\rangle$$

for $p^0 > -mc$, and

$$\left[-\frac{p^{02}}{2m}, \infty\right\rangle$$

for $p^0 < -mc$.

To prove this statement we will consider separately the discrete and continuous spectra.

(*i*) *The Discrete Spectrum*. Let us pose

$$\gamma = \left[(l+1/2)^2 - \alpha^2\right]^{1/2} - 1/2$$

$$\mu^2 = -4\frac{p^{02} + 2me(p^0)}{\hbar^2} > 0$$

$$\lambda = 2\frac{mc\alpha}{\mu\hbar}\left(1 + \frac{p^0}{mc}\right), \qquad \mu > 0$$

$$\rho = \mu r$$

Then, (9) can be written

$$\left[\partial_\rho^2 - \frac{(l+1/2)^2 - 1/4}{\rho^2} + \frac{\lambda}{\rho} - \frac{1}{4}\right] g(\rho) = 0$$

$$g(\rho) \in L^2(\mathbb{R}^+, d\rho) \text{ and } g(0) = 0 \quad (10)$$

The equation (10) is the Whittaker equation, and the solutions to the above problem can be expressed in terms of confluent hypergeometric functions M, i.e.,

$$g_{\gamma\lambda}(\rho) = e^{-\rho/2}\rho^{\gamma+1}M(\gamma + 1 - \lambda, 2\gamma + 2, \rho)$$

(13.1.31, 13.1.32, Abramowitz and Stegun, 1966). The condition $g \in L^2(\mathbb{R}^4, dr)$ implies that

$$\gamma + 1 - \lambda = -\nu = 0, 1, 2, \ldots$$

Thus,

$$e_{nl}(p^0) = -\frac{1}{2}\frac{(1 + p^0/mc)^2\alpha^2 mc^2}{\left\{n - l - 1/2 + \left[(l+1/2)^2 - \alpha^2\right]^{1/2}\right\}^2} - \frac{p^{02}}{2m}$$

with $n = \nu + l + 1$, expresses the discrete spectrum of $\hat{h}_{\sigma^{-1}}(p^0)$.

(*ii*) *The Continuous Spectrum*. Let us pose

$$k^2 = p^{02} + 2me(p^0) > 0$$

$$\kappa = \frac{mc\alpha}{k}\left(1 + \frac{p^0}{mc}\right), \qquad k > 0$$

$$\rho = \kappa r \tag{11}$$

Then, (9) can be written

$$\left[\partial_\rho^2 - \frac{(\gamma+1)^2 - 1/4}{\rho^2} + \frac{\kappa}{\rho} + \frac{1}{4}\right] g(\rho) = 0$$

$$g(0) = 0$$

Accordingly by posing $\rho = \mp iz$ we again obtain the Whittaker equation. Thus the solutions are of the form

$$g(\rho) = e^{\pm i\rho/2}\rho^{\gamma+1} M(\gamma + 1 \mp i\kappa, 2\gamma + 2, \mp i\rho)$$

and according to (11)

$$e(p^0) = \frac{k^2 - p^{02}}{2m} \in \left[-\frac{p^{02}}{2m}, \infty\right\rangle$$

4.3. The Internal Energy Spectrum. From the preceding results we can conclude that $H = H_D \otimes H_C$, where

$$H_D = \int_{-M}^{\infty} L^2\left(\mathbb{R}^3, \int_{-(m_2/M)(\Delta M + M)c}^{(m_1/M)(\Delta M + M)c} \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \mathbb{C}^{2l+1} \chi_{p^0 > -mc}(p^0)\, dp^0;\right.$$

$$\left.\frac{cd^3P}{\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2}}\right) d\Delta M$$

$$H_C \cong \int_{-M}^{\infty} L^2\left(\mathbb{R}^3, \int_{(-m_2/M)(\Delta M + M)c}^{(m_1/M)(\Delta M + M)c} L^2\left(\left[-\frac{p^{02}}{2m}, \infty\right\rangle, \sum_{l=0}^{\infty} \mathbb{C}^{2l+1}; de\right) dp^0;\right.$$

$$\left.\frac{cd^3P}{\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2}}\right) d\Delta M$$

with

$$\chi_{p^0 > -mc}(p^0) = \begin{cases} 1 & \text{if } p^0 > -mc \\ 0 & \text{if } p^0 < -mc \end{cases}$$

defines a representation diagonalizing $\hat{h}$ and $(\hat{h}^i)^2$.

The complementary subspaces H_D and H_C, i.e., the subspaces of the discrete and continuous spectrum of $\hat{h}$, have the following canonical representations also:

$$H_D \cong L^2\left(N, \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \mathbb{C}^{2l+1}; \frac{d\Delta M c\, d^3P \chi_{p^0 > -mc}(p^0)\, dp^0}{\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2}}\right)$$

$$\cong L^2\left(N, \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \mathbb{C}^{2l+1}; d^3P \chi_{p^0 > -mc}(p^0)\, dp^0\right) \tag{12}$$

and

$$H_C \cong \int_{-M}^{\infty} L^2\left(\mathbb{R}^3, \int_{-(m_2/M)(\Delta M + M)c}^{(m_1/M)(\Delta M + M)c} L^2\left(\mathbb{R}^+, \sum_{l=0}^{\infty} \mathbb{C}^{2l+1}; dk\right) dp^0;\right.$$

$$\left.\frac{c\, d^3P}{((P^i)^2 + (\Delta M + M)^2 c^2)^{1/2}}\right) d\Delta M$$

$$\cong \int_{-M}^{\infty} L^2\left(\mathbb{R}^3, \int_{-(m_2/M)(\Delta M + M)c}^{(m_1/M)(\Delta M + M)c} L^2(\mathbb{R}^3; d^3k)\, dp^0;\right.$$

$$\left.\frac{c\, d^3P}{\left[(P^i)^2 + (\Delta M + M)^2 c^2\right]^{1/2}}\right) d\Delta M$$

$$\cong L^2(N \times \mathbb{R}^3; d^4P\, dp^0\, d^3k)$$

with N as in Definition 2.1. Denote by $\hat{h}_D$ and $\hat{h}_C$ the images of $\hat{h}$ under the projections onto H_D and H_C, respectively; then,

$$(\hat{h}_D f^D)_{nl}(P^\mu, p^0) = \left(-\frac{1}{2} \frac{(1+p^0/mc)^2 \alpha^2 mc^2}{\left\{ n - l - \frac{1}{2} + \left[(l+\frac{1}{2})^2 - \alpha^2 \right]^{1/2} \right\}^2} - \frac{p^{02}}{2m} \right) \times f_{nl}^D(P^\mu, p^0)$$

$$(\hat{h}_C f^C)(P^\mu, p^0, k^i) = \frac{(k^i)^2 - p^{02}}{2m} f^C(P^\mu, p^0, k^i)$$

Moreover, the representation $\hat{U}$ of $SO(1,3)_0 \underset{s}{\times} \mathbb{R}^4$ (Definition 3.1) induces the representations

$$(\hat{U}_D(\Lambda(\theta^i, u^i), a^\mu) f^D)_{tnl}(P^\mu, p^0) = \exp\left(-\frac{i}{\hbar} P_\mu a^\mu + \frac{i}{\hbar} P_\mu v^\mu(-u^i) t \right) \times D^{(l)}(\Lambda(\varphi_w^i(P^\mu, \Lambda(\theta^i, u^i)))) \times f_{tnl}^D(\Lambda^{-1}(\theta^i, u^i)^\mu_\nu P^\nu + M v^\mu(-u^i))$$

and

$$(\hat{U}_C(\Lambda(\theta^i, u^i), a^\mu) f^C)_t(P^\mu, p^0, k^i) = \exp\left(-\frac{i}{\hbar} P_\mu a^\mu + \frac{i}{\hbar} P_\mu v^\mu(-u^i) t \right) f_t\Big(\Lambda^{-1}(\theta^i, u^i)^\mu_\nu P^\nu + M v^\mu(-u^i), p^0, \Lambda^{-1}(\varphi_w^i(P^\mu, \Lambda^{-1}(\theta^i, u^i)))^i_j k^j \Big)$$

in H_D and H_C, respectively. $\hat{D}^{(l)}$ is an irreducible unitary representation of $SO(3)$ in $\mathbb{C}^{2l+1}$.

The Discrete Internal Energy Spectrum. In the representation (12) diagonalizing $\Delta\hat{M}_D$, $\hat{p}_D^0$, and $\hat{h}_D$, $\hat{\alpha}_D$ and $\hat{\beta}_D$ are represented by

$$(\hat{\alpha}_D g^D)_{nl}(\Delta M, p^0) = \left\{ \Delta M + \frac{\Delta M^2}{2M} - e_{nl}(p^0) \right\} g_{nl}^D(\Delta M, p^0)$$

$$(\hat{\beta}_D g^D)_{nl}(\Delta M, p^0) = \left\{ p^0 - \frac{1}{c}\left(1 - 4\frac{m}{M}\right)^{1/2} \frac{e_{nl}(p^0)}{[1 + 2e_{nl}(p^0)/Mc^2]^{1/2}} \right\} \times g_{nl}^D(\Delta M, p^0)$$

with

$$e_{nl}(p^0) = -\frac{1}{2}\frac{(1+p^0/mc)^2\alpha^2mc^2}{\left\{n-l-\frac{1}{2}+\left[(l+\frac{1}{2})^2-\alpha^2\right]^{1/2}\right\}^2}$$

According to our definition, the internal energy spectrum of the system is

$$e_{nl}^{(0)}(p^0) = e_{nl}(p^0(\beta))|_{\beta=0}$$

Thus, consider first the case where $m_1 \to \infty$, i.e., the system of a particle of mass m in an external Coulomb field. Then,

$$\hat{\beta}_{nl}(p^0) = p^0 - \frac{1}{c}e_{nl}(p^0)$$

and accordingly, we obtain the well-known expression

$$e_{nl}^{(0)} = mc^2\left(1+\alpha^2\left\{n-l-\frac{1}{2}+\left[(l+\frac{1}{2})^2-\alpha^2\right]^{-1/2}\right\}^{-1/2}\right)-mc^2$$

$$\sim -\frac{1}{2}\frac{\alpha^2}{n^2}mc^2+\frac{1}{2}\frac{\alpha^4}{n^4}\left(\frac{3}{4}-\frac{n}{l+\frac{1}{2}}\right)mc^2$$

which also gives the energy levels predicted by the Klein–Gordon equation with the Coulomb potential (Shiff, 1955). In the general case, we get to the order α^4

$$e_n^{(0)} \sim -\frac{1}{2}\frac{\alpha^2}{n^2}mc^2+\frac{1}{2}\frac{\alpha^4}{n^4}\left[\left(1-4\frac{m}{M}\right)^{1/2}-\frac{1}{4}\left(1-4\frac{m}{M}\right)-\frac{n}{l+\frac{1}{2}}\right]mc^2$$

The eigenfunctions of $\hat{h}^{(0,0)}$ are functions of the form $f(P^i)\times\tilde{g}_{nl}(r,\phi,\theta)$ in

$$H_D^{(0,0)} = \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}L^2\left(\mathbb{R}^3,\mathbb{C}^{2l+1};\frac{cd^3P}{\left[(P^i)^2+(\Delta M+M)^2c^2\right]^{1/2}}\right)$$

with

$$\tilde{g}_{nl}(r,\theta,\phi) = \left[\mu(p^0(\beta))\right]^{1/2}\left[\partial_{p^0}\beta(p^0(\beta))\right]^{-1/2}$$
$$\times\frac{1}{r}g_{n-l,[(l+1/2)^2-\alpha^2]^{1/2}-1/2}\left[\mu(p^0(\beta))r\right]Y^m(\theta,\phi)|_{\beta=0}$$

Thus it follows that when the system is in a bound state of internal energy $e_{nl}^{(0)}$, the center of mass appears as a free particle of spin l and effective mass $M' = m(1+2e_{nl}^{(0)}/Mc^2)^{1/2}$.

The Continuous Internal Energy Spectrum. The spectrum of $h_C^{(0,0)}$ is given by

$$e_k^{(0)} = e\left(p^0(\beta)\right)|_{\beta=0}$$
$$= \frac{2m}{M}\frac{(k^i)^2}{2m} + mc^2\left(\left\{1+2\frac{M-2m}{mMc^2}\frac{(k^i)^2}{2m} + \frac{4}{M^2c^4}\left[\frac{(k^i)^2}{2m}\right]^2\right\}^{1/2} - 1\right)$$

Thus for the total energy in the center-of-mass frame of reference we get

$$\Delta M_C^{(0,0)}c^2 = c\left[(k^i)^2 + m_1^2c^2\right]^{1/2} - m_1c^2 + c\left[(k^i)^2 + m_2^2c^2\right]^{1/2} - m_2c^2$$

On the other hand, following the discussion of Section 3.4, one shows that the asymptotic scattering system is equivalent to the system of two free particles.

5. DISCUSSION

There are no essential formal difficulties involved in the construction of this theory. It is in every respect mathematically well defined, and moreover, satisfies a number of a priori conditions which, having been suggested by the Galilean theory, seem to be natural to impose in the Einstein relativistic framework also. The most notable of these conditions is the universality of the definition of a particle. In fact, it seems that this condition, together with the assumptions that the theory should incorporate the usual Einstein relativistic kinematics and the property of relative localization in space-time, fixes the structure of the theory.

It is clear that a general theory like the present one can not be directly falsified; it contains sufficiently phenomenological input to describe in a proper way the kinematical behaviour of a system of Einstein relativistic particles, and thus, any disagreement between the predictions of a given model formulated in this theory and experience can be referred to the particular model being considered. In fact, a theory like this can only be judged on the basis of criteria of usefulness and applicability, i.e., according to whether it can be used to formulate workable models of physical systems exhibiting features which one would characterize as being Einstein relativistic. It is with such a judgement in mind that we have considered the system of two gravitating classical particles (Aaberge, 1979), and the system of two charged quantal particles of spin 0, the results obtained being such as to give a partial justification of the theory. None of these models describe systems which are properly speaking Einstein relativistic. In fact, the Einstein relativistic effects are far from dominating. They introduce only small refinements to the Galilean models. There exists however, a simple model of hadronic scattering (Schrempp and Schrempp, 1980) which can be considered as an example of potential scattering in our theory. This model does work surprisingly well also in the high-energy domain, thus supporting an extension of the applicability of our theory to this domain also.

A priori, possible applications of this theory would be to the description of hydrogenlike systems, exotic atoms and meson systems via the quark model. In particular, the nonrelativistic models of the Υ/ψ mesons (Quigg and Rosner, 1979; Grosse and Martin, 1980) can directly be taken over in the present theory, since $m_1 = m_2$ and $\beta = 0 \Rightarrow p^0 = 0$, means that the relativistic internal Hamiltonian equals the nonrelativistic one.

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REFERENCES

Aaberge, T. (1975). *Helvetica Physica Acta*, **48**, 173.

Aaberge, T. (1977). *Helvetica Physica Acta*, **50**, 331.

Aaberge, T. (1979). *General Relativity and Gravitation*, **10**, 897.

Aaberge, T. (1982). *International Journal of Theoretical Physics*, **22**, 851.

Abramowitz, M., and Stegun, J. A. (1966). *Handbook of Mathematical Functions*. National Bureau of Standards Applied Mathematics Series 55, Washington, DC.

Aghassi, J. J., Roman, P., and Santilli, R. M. (1970). *Physical Review D*, **1**, 2753.

Bakamjian, B., and Thomas, L. H. (1953). *Physical Review*, **92**, 1300.

Cartan, E. (1971). *Leçons sur les invariants intégreaux*, Hermann, Paris.

Coester, F. (1965). *Helvetica Physica Acta*, **38**, 7.

Fong, R., and Sucher, J. (1964). *Journal of Mathematical Physics*, **5**, 456.

Grosse, H., and Martin, A. (1980). *Reports on Physics*, **60**, 341.

Horwitz, L. P., and Piron, C. (1973). *Helvetica Physica Acta*, **46**, 316.

Mackey, G. W. (1968). *Induced Representations of Groups and Quantum Mechanics*, W. A. Benjamin Inc., New York.

Meetz, K. (1964). *Il Nuovo Cimento* **34**, 690.

Newton, T. D. and Wigner, E. P. *Reviews of Modern Physics* **21**, 400.

Pearle, P. M. (1968). *Physical Review*, **168**, 1429.

Piron, C. (1976). *Foundations of Quantum Physics*, W. A. Benjamin Inc., Reading, Massachusetts.

Quigg, C., and Rosner, J. L. (1979). *Reports on Physics*, **56**, 167.

Reuse, F. (1979). *Foundations of Physics*, **9**, 865.

Schiff, L. I. (1955). *Quantum Mechanics* (Paragraph 4.2), McGraw Hill, New York.

Schrempp, B., and Schrempp, F. (1980). *Nuclear Physics C*, **163**, 397.